COMP2111 Week 4
Term 1, 2024
Propositional Logic II

Summary of topics

- Well-formed formulas
- Boolean Algebras
- Valuations
- CNF/DNF
- Proof
- Natural deduction

Terminology and Rules

We'll look at two normal forms of propositional formulae:

- Conjunctive normal form (CNF)

  "a conjunction of disjunctions of possibly negated atoms"

  Example: $a \wedge (b \vee \neg c) \wedge (a \vee c \vee d)$

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  Example: $a \wedge (b \vee \neg c) \wedge (a \vee c \vee d)$

- Disjunctive normal form (DNF)

  "a disjunction of conjunctions of possibly negated atoms"

  Example: $(a \wedge b) \vee (\neg a \wedge \neg b)$

Terminology and Rules

For readability, we write $\overline{\varphi}$ for $\neg\varphi$. A **literal** is an expression p or $\overline{p}$, where p is a propositional atom.

A formula is in CNF if it has the form

$$C_0 \wedge C_1 \wedge C_2 \wedge \cdots \wedge C_n$$

where each **clause** C_i is a disjunction of literals e.g. $p \vee q \vee \overline{r}$.

A propositional formula is in DNF if it has the form

$$D_0 \vee D_1 \vee D_2 \vee \cdots \vee D_n$$

where each clause D_i is a conjunction of literals e.g. $p \wedge q \wedge \overline{r}$.

Motivation

- Checking satisfiability in DNF is straightforward.
- Checking validity in CNF is straightforward.
- Every propositional formula can be rewritten into DNF/CNF form (but it's computationally expensive)

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  Is there a clause that doesn't have two contradictory literals? I.e., doesn't have both p and $\overline{p}$ for some p? If yes, it's satisfiable.

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- Checking validity in CNF is straightforward.

  Does every clause have contradictory literals? If yes, it's valid.

- Every propositional formula can be rewritten into DNF/CNF form (but it's computationally expensive)

Theorem

For every propositional formula ϕ, there is an equivalent formula in CNF and an equivalent formula in DNF.

We will show two methods of obtaining the CNF/DNF:

- An algorithm for rewriting formulas.
- Reading them off the truth table.

Rewriting to CNF/DNF

1. Eliminate all $\rightarrow$ and $\leftrightarrow$, using

$$A \rightarrow B \equiv \overline{A} \vee B \qquad A \leftrightarrow B \equiv (A \wedge B) \vee (\overline{A} \wedge \overline{B})$$

2. Push negations inwards, using **De Morgan** and **double negation**.
3. Push disjunctions inwards (for CNF), or conjunctions inwards (for DNF), using **distributivity**.

Step 2: Push Negations Down

Using **De Morgan's** laws and the **double negation** rule

$$\overline{x \vee y} \equiv \overline{x} \wedge \overline{y}$$
$$\overline{x \wedge y} \equiv \overline{x} \vee \overline{y}$$
$$\overline{\overline{x}} \equiv x$$

we push negations down towards the atoms, until we obtain a formula that is formed from literals using only $\wedge$ and $\vee$.

Step 3: Use Distribution to Convert to CNF

Using the distributivity rules (until we can't):

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$
$$(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$$

we obtain a CNF formula. For a DNF formula, use the dual rules.

Step 2:

Example

$$\begin{aligned}\overline{p(\overline{rs} \vee q)} &\equiv \overline{\overline{p}} \vee \overline{\overline{rs} \vee q} \\ &\equiv p \vee \overline{\overline{rs}} \wedge \overline{q} \\ &\equiv p \vee rs\overline{q}\end{aligned}$$

Step 3:

Example

$$\begin{aligned}p \vee rs\overline{q} &\equiv (p \vee r)(p \vee s\overline{q}) \\ &\equiv (p \vee r)(p \vee s)(p \vee \overline{q}) \qquad \text{CNF}\end{aligned}$$

Canonical Form DNF, informally

Extract a canonical DNF E^{dnf} from a truth table by making one disjunct for every row where E holds:

x	y	E
F	F	T
F	T	F
T	F	T
T	T	T

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x	y	E	
F	F	T	$\overline{x} \wedge \overline{y}$
F	T	F	
T	F	T	
T	T	T	

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F	T	F	
T	F	T	$x \wedge \overline{y}$
T	T	T	

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F	T	F	
T	F	T	$x \wedge \overline{y}$
T	T	T	$x \wedge y$

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Extract a canonical DNF E^{dnf} from a truth table by making one disjunct for every row where E holds:

x	y	E	
F	F	T	$\overline{x} \wedge \overline{y}$
F	T	F	
T	F	T	$x \wedge \overline{y}$
T	T	T	$x \wedge y$

thus $E^{dnf} = (\overline{x} \wedge \overline{y}) \vee (x \wedge \overline{y}) \vee (x \wedge y)$

Note that this can be simplified to $\overline{y} \vee (x \wedge y)$ or $(\overline{x} \wedge \overline{y}) \vee x$

Canonical Form DNF, formally

Given a Boolean expression E, we can construct an equivalent DNF E^{dnf} from the lines of the truth table where E is true: Given an assignment v from $\{x_1 \ldots x_i\}$ to $\mathbb{B}$, define the literal

$$\ell_i = \begin{cases} x_i & \text{if } v(x_i) = \mathtt{true} \\ \overline{x_i} & \text{if } v(x_i) = \mathtt{false} \end{cases}$$

and a product $t_v = \ell_1 \wedge \ell_2 \wedge \ldots \wedge \ell_n$.

Example

If $v(x_1) = \mathtt{true}$ and $v(x_2) = \mathtt{false}$ then $t_v = x_1 \wedge \overline{x_2}$

The **canonical DNF** of E is

$$E^{dnf} = \bigvee_{[\![E]\!]_v = \mathtt{true}} t_v$$

Canonical Form CNF, informally

Extract a canonical CNF by making one conjunct for every row where E does not hold, *negating the atoms*:

x	y	E
F	F	F
F	T	F
T	F	T
T	T	F

Canonical Form CNF, informally

Extract a canonical CNF by making one conjunct for every row where E does not hold, *negating the atoms*:

x	y	E	
F	F	F	$x \vee y$
F	T	F	
T	F	T	
T	T	F	

Canonical Form CNF, informally

Extract a canonical CNF by making one conjunct for every row where E does not hold, *negating the atoms*:

x	y	E	
F	F	F	$x \vee y$
F	T	F	$x \vee \overline{y}$
T	F	T	
T	T	F	

Canonical Form CNF, informally

Extract a canonical CNF by making one conjunct for every row where E does not hold, *negating the atoms*:

x	y	E	
F	F	F	$x \vee y$
F	T	F	$x \vee \overline{y}$
T	F	T	
T	T	F	$\overline{x} \vee \overline{y}$

Canonical Form CNF, informally

Extract a canonical CNF by making one conjunct for every row where E does not hold, *negating the atoms*:

x	y	E	
F	F	F	$x \vee y$
F	T	F	$x \vee \overline{y}$
T	F	T	
T	T	F	$\overline{x} \vee \overline{y}$

thus $E^{cnf} = (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{x} \vee \overline{y})$

Canonical CNF

After pushing negations down, the negation of a DNF is a CNF (and vice versa).

$\Rightarrow$ Given an expression E, we can obtain an equivalent CNF by finding a DNF for $\neg E$ and then applying De Morgan's laws.

$\Leftrightarrow$ Look at rows in the truth table of E that contain `false` and *negate* the literals.

Summary of topics

- Well-formed formulas
- Boolean Algebras
- Valuations
- CNF/DNF
- Proof
- Natural deduction

Motivation

(Informal) proofs are hard to check. Why?

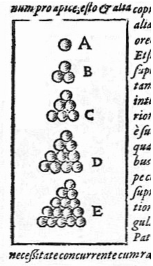
Most mathematicians don't explain all the details when they write proofs; the idea is that an expert reader can fill in the gaps.

"Without loss of generality, we may assume ...", "It is easily seen that ..."

Checking such proofs take significant time and expertise.

Motivation

In 1998,



Tom Hales proved the Kepler conjecture

in 300 pages of mathematical prose and $40k$ lines of (unverified) code. He submitted his work to a journal for peer review.

Motivation

Seven years later, the proof is accepted. A journal editor describes the process:

> *The referees put a level of energy into this that is, in my experience, unprecedented. They ran a seminar on it for a long time. A number of people were involved, and they worked hard. They checked many local statements in the proof, and each time they found that what you claimed was in fact correct. Some of these local checks were highly non-obvious at first, and required weeks to see that they worked out . . . They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem.*

Motivation

Formal proofs are *easy* to check.

"Easy" as in: completely mechanical, requires no domain expertise, can be outsorced to a computer program—just check that each step follows the rules.

"The rules": a precise, unambiguous description of which proof steps are allowed. Different formalisms may have different rules; we'll study one (natural deduction).

Given a theory T and a formula φ, how do we show $T \models \varphi$?

- Consider all valuations v (SEMANTIC approach)
- Use a sequence of **inference rules** to show that φ is a logical consequence of T (SYNTACTIC approach)

New notation

We will distinguish between two concepts:

Notation	Name	Meaning
$T \models \varphi$	entailment	whenever T is true, φ is true.
$T \vdash \varphi$	provability	we can prove φ from T using the inference rules.

Why does this distinction matter?

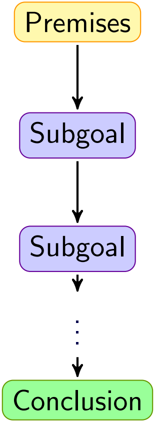
New notation

Why distinguish truth from provability?

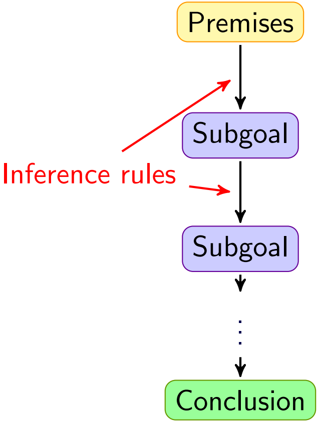
If our inference rules are poorly chosen, they could "prove" things that are not true.

Conversely, something might be true, but our inference rules might not be powerful enough to prove it.

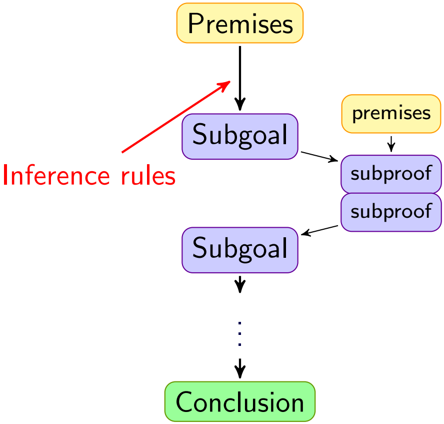
Proof stucture



Proof stucture



Proof stucture



Inference rules

In its simplest form, an **inference rule** is a statement of the form:

If I have a proof of this then I have a proof of that

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If I have a proof of this then I have a proof of that

A more complicated form:
If I have a proof of this (under those assumptions)
then I have a proof of that (under these* assumptions)

NB

The sets of assumptions need not be the same!

Inference rules

In its simplest form, an **inference rule** is a statement of the form:

If I have a proof of this then I have a proof of that

Yet more complicated form:

If I have a proof of this (under those assumptions)
and I have a proof of this (under those assumptions)
and ...
then I have a proof of that (under these assumptions)

NB

The sets of assumptions need not be the same!

Inference rules: notation

So an inference rule is a statement of the form:

If $T_1 \vdash \varphi_1$ and $T_2 \vdash \varphi_2$ and ... and $T_n \vdash \varphi_n$ then $T \vdash \psi$

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Alternative notation:

$$\frac{T_1 \vdash \varphi_1 \qquad T_2 \vdash \varphi_2 \qquad \cdots \qquad T_n \vdash \varphi_n}{T \vdash \psi}$$

Inference rules: notation

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Alternative notation:

$$\frac{\varphi_1 \qquad \varphi_2 \qquad \cdots \qquad \varphi_n}{\psi}$$

(If $T_1 = T_2 = \ldots = T$)

Inference rules: notation

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If $T_1 \vdash \varphi_1$ and $T_2 \vdash \varphi_2$ and ... and $T_n \vdash \varphi_n$ then $T \vdash \psi$

Alternative notation:

$$\frac{\varphi_1 \qquad \begin{array}{c}[\alpha]\\ \vdots \\ \varphi_2\end{array} \qquad \cdots \qquad \varphi_n}{\psi}$$

(If $T_2 = T \cup \{\alpha\}$)

Inference rules: examples

$\wedge$-elimination:

$$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

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$\rightarrow$-elimination:
(Modus Ponens)

$$\frac{A \rightarrow B \qquad A}{B}\ (\rightarrow\text{-E})$$

Inference rules: examples

$\wedge$-elimination:

$$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

$\rightarrow$-elimination:
(Modus Ponens)

$$\frac{A \rightarrow B \qquad A}{B}\ (\rightarrow\text{-E})$$

$\rightarrow$-introduction:

$$\begin{array}{c}[A]\\ \vdots \\ \dfrac{B}{A \rightarrow B}\ (\rightarrow\text{-I})\end{array}$$

Proof layout

We will look at three alternative ways to present proofs:

Tabular	Easy to typeset, annoying to read.
Fitch-style	Easy to read, annoying to typeset.
Proof trees	Nice for short proofs, very unwieldy for long proofs.

Tabular

Line	Premises	Formula	Rule	References

Tabular

Example:

$$\text{Prove:} \quad A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$$

Line	Premises	Formula	Rule	References
1		$A \rightarrow B$	Premise	
2		$B \rightarrow C$	Premise	

Tabular

Example:

Prove: $A \to B, B \to C \vdash A \to C$

Line	**Premises**	**Formula**	**Rule**	**References**
1		$A \to B$	Premise	
2		$B \to C$	Premise	
	$1, 2$	$A \to C$		

Tabular

Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	

Tabular

Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$

Tabular

Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$
5	$1, 2, 3$	C	$\to$ -E	$2, 4$

Tabular

Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$
5	$1, 2, 3$	C	$\to$ -E	$2, 4$
6	$1, 2$	$A \to C$	$\to$ -I	5

Premises

You may assume anything, but it may not be helpful!
You must **discharge** any assumptions you make along the way.

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You must **discharge** any assumptions you make along the way.

For example, let's "prove" $A \to B \vdash B$:

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		A	Premise	
3	$1, 2$	B	$\to$ -E	$1, 2$

Now we have no way of discharging A. Thus we ended up proving this instead:

$$A \to B, A \vdash B$$

Proof tree layout

Upside-down tree (root at bottom):

- Natural structure arising from rule syntax
- Premises at leaves
- Conclusion at root

$$\dfrac{\dfrac{B \to C \qquad \dfrac{A \to B \qquad [A]}{B}\;(\to\text{-E})}{C}\;(\to\text{-E})}{A \to C}\;(\to\text{-I})$$

Other proof layouts: Fitch-style

- Style used in online tool
- Premises above the notch, subgoals below.
- Subproofs are indented

1.	$A \to B$	
2.	$B \to C$	
3.	A	
4.	B	$\to$-E: 1,3
5.	C	$\to$-E: 2,4
6.	$A \to C$	$\to$-I: 3–5

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Online resources

Open logic project (https://proofs.openlogicproject.org/)

- Freely available textbooks
- Online proof checker
- Sample exercises

Natural deduction

Proof system intended to mirror "natural reasoning":

- No axioms (no inherent truths)
- 15 inference rules: primarily grouped into pairs (and trios) of rules tasked with **introducing** and **eliminating** boolean operators from the chain of reasoning.

Operator	Introduction	Elimination
$\wedge$	$\wedge$-I	$\wedge$-E1 $\wedge$-E2
$\vee$	$\vee$-I1 $\vee$-I2	$\vee$-E
$\rightarrow$	$\rightarrow$-I	$\rightarrow$-E
$\leftrightarrow$	$\leftrightarrow$-I	$\leftrightarrow$-E1 $\leftrightarrow$-E2
$\neg$	$\neg$-I	$\neg$-E IP
$\perp$	$\neg$-E	X

$\wedge$ Introduction and Elimination

$\wedge$-introduction:

$$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{-I})$$

$\wedge$ Introduction and Elimination

$\wedge$-introduction: $$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{-I})$$

$\wedge$-elimination (1): $$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

$\wedge$-elimination (2): $$\frac{A \wedge B}{B}\ (\wedge\text{-E2})$$

$\vee$ Introduction and Elimination

$\vee$-introduction (1): $$\frac{A}{A \vee B}\,(\vee\text{-I1})$$

$\vee$-introduction (2): $$\frac{B}{A \vee B}\,(\vee\text{-I1})$$

$\vee$ Introduction and Elimination

$\vee$-introduction (1):

$$\frac{A}{A \vee B}\ (\vee\text{-I1})$$

$\vee$-introduction (2):

$$\frac{B}{A \vee B}\ (\vee\text{-I1})$$

$\vee$-elimination:

$$\frac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}\ (\vee\text{-E})$$

$\rightarrow$ Introduction and Elimination

$\rightarrow$-introduction:

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B}\ (\rightarrow\text{-I})$$

$\rightarrow$-elimination:
(Modus Ponens)

$$\dfrac{A \rightarrow B \qquad A}{B}\ (\rightarrow\text{-E})$$

$\leftrightarrow$ Introduction and Elimination

$\leftrightarrow$-introduction:

$$\frac{\begin{array}{cc}[A] & [B] \\ \vdots & \vdots \\ B & A\end{array}}{A \leftrightarrow B}\ (\leftrightarrow\text{-I})$$

$\leftrightarrow$-elimination (1):

$$\frac{A \leftrightarrow B \qquad A}{B}\ (\leftrightarrow\text{-E1})$$

$\leftrightarrow$-elimination (2):

$$\frac{A \leftrightarrow B \qquad B}{A}\ (\leftrightarrow\text{-E1})$$

¬ Introduction and Elimination and Indirect Proof

¬-introduction:

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\ (\neg\text{-I})$$

¬-elimination:
(⊥-introduction)

$$\dfrac{A \qquad \neg A}{\bot}\ (\neg\text{-E})$$

$\neg$ Introduction and Elimination and Indirect Proof

$\neg$-introduction:

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\ (\neg\text{-I})$$

$\neg$-elimination:
($\bot$-introduction)

$$\dfrac{A \qquad \neg A}{\bot}\ (\neg\text{-E})$$

Indirect proof:

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots\\ \bot\end{array}}{A}\ (\text{IP})$$

Explosion

Explosion:
($\perp$-elimination)

$$\frac{\perp}{A}\,(X)$$

Derived rules

Several useful rules are available in the proof checker. Feel free to use them in assignments and exam, unless otherwise stated.

Double negation elimination:

$$\frac{\neg\neg A}{A}\ \text{(DNE)}$$

Reiteration:

$$\frac{A}{A}\ \text{(R)}$$

Law of excluded middle:

$$\frac{\begin{array}{cc}[A] & [\neg A] \\ \vdots & \vdots \\ B & B\end{array}}{B}\ \text{(LEM)}$$

Derived rules

Disjunctive syllogism: $$\frac{A \vee B \qquad \neg A}{B}\ (\mathrm{DS})$$

Modus Tollens: $$\frac{A \to B \qquad \neg B}{\neg A}\ (\mathrm{MT})$$

De Morgans Laws (e.g.): $$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}\ (\mathrm{DM})$$

Soundness and completeness

Two key properties of a “good” system:

Soundness: The system only proves valid statements: If $T \vdash \varphi$ then $T \models \varphi$.

Completeness: The system can prove any valid statement: If $T \models \varphi$ then $T \vdash \varphi$.

NB

An unsound system is kinda useless.

An incomplete system can still be useful. For most logics, no sound and complete system exists (c.f. Gödel’s incompleteness theorem).

Soundness and completeness

Theorem

Natural deduction is sound and complete. That is,

$$T \vdash \varphi \quad \textit{if and only if} \quad T \models \varphi$$

Soundness and completeness

Theorem

Natural deduction is sound and complete. That is,

$$T \vdash \varphi \quad \textit{if and only if} \quad T \models \varphi$$

Corollary

The following are equivalent:

- $\varphi_1, \varphi_2, \ldots, \varphi_n \models \varphi$
- $\varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \varphi$ *is a tautology*
- $\varphi_1 \rightarrow (\varphi_2 \rightarrow (\cdots \rightarrow (\varphi_n \rightarrow \varphi)) \cdots)$ *is a tautology*
- $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \varphi$
- $\varphi_1, \varphi_2, \ldots, \varphi_{n-1} \vdash \varphi_n \rightarrow \varphi$
- *(and so on)*

This guy again

Tom Hales, evidently unsatisfied that his peer reviewers were only 99% convinced that he'd proven the Kepler conjecture, decided to construct a *formal* proof.

In 2014, after a decade-long slog, his team of around 20 mathematicians, computer scientists and engineers produced a fully formal proof, with every last inference machine-checked to follow the inference rules of HOL.

The Kepler conjecture was settled.

Credits

 This picture ©2012 Slawekb, CC BY-SA 3.0

The journal editor's quote can be found here:

- Hales, Thomas C. *Formal Proof*. Page 1370–1380 of *Notices of the AMS*, volume 55, number 11. American Mathematical Society 2008.

Summary of topics

- Well-formed formulas
- Boolean Algebras
- Valuations
- CNF/DNF
- Proof
- Natural deduction
- Bonus examples

Proof examples

What follows is a bunch of example derivations in natural deduction, with all three styles well represented.

I do not plan to go over these in the lecture (favouring instead live demos).

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1
6	1	$B \wedge C$	$\wedge$-I	4, 5

Proof example

Prove: $(A \land B) \land C \vdash A \land (B \land C)$

Line	Premises	Formula	Rule	References
1		$(A \land B) \land C$	Premise	
2	1	$A \land B$	$\land$-E1	1
3	1	A	$\land$-E1	2
4	1	B	$\land$-E2	2
5	1	C	$\land$-E2	1
6	1	$B \land C$	$\land$-I	4, 5
7	1	$A \land (B \land C)$	$\land$-I	3, 6

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. (A ∧ B) ∧ C	
2. A ∧ B	∧-E1: 1

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$	
2. $A \wedge B$	$\wedge$-E1: 1
3. A	$\wedge$-E1: 2

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. (A ∧ B) ∧ C	
2. A ∧ B	∧-E1: 1
3. A	∧-E1: 2
4. B	∧-E2: 2

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$	
2. $A \wedge B$	$\wedge$-E1: 1
3. A	$\wedge$-E1: 2
4. B	$\wedge$-E2: 2
5. C	$\wedge$-E2: 1

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. (A ∧ B) ∧ C	
2. A ∧ B	∧-E1: 1
3. A	∧-E1: 2
4. B	∧-E2: 2
5. C	∧-E2: 1
6. B ∧ C	∧-I: 4,5

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$	
2. $A \wedge B$	$\wedge$-E1: 1
3. A	$\wedge$-E1: 2
4. B	$\wedge$-E2: 2
5. C	$\wedge$-E2: 1
6. $B \wedge C$	$\wedge$-I: 4,5
7. $A \wedge (B \wedge C)$	$\wedge$-I: 3,6

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$(A \wedge B) \wedge C$$

$$(A \wedge B) \wedge C \qquad\qquad A \wedge B \qquad\qquad (A \wedge B) \wedge C$$

$$A \wedge B \qquad\qquad\qquad B \qquad\qquad\qquad C$$

$$A \qquad\qquad\qquad\qquad B \wedge C$$

$$A \wedge (B \wedge C)$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$
\begin{array}{ccccc}
 & & (A \wedge B) \wedge C & & \\
\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1}) & & A \wedge B & & (A \wedge B) \wedge C \\
 & & B & & C \\
A & & & B \wedge C & \\
 & A \wedge (B \wedge C) & & &
\end{array}
$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$(A \wedge B) \wedge C$$

$$\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{A}\,(\wedge\text{-E1}) \qquad \begin{array}{c} A \wedge B \\ B \end{array} \qquad \begin{array}{c} (A \wedge B) \wedge C \\ C \end{array}$$

$$B \wedge C$$

$$A \wedge (B \wedge C)$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\;(\wedge\text{-E1})}{A}\;(\wedge\text{-E1})
\qquad
\begin{array}{c}
\dfrac{(A \wedge B) \wedge C}{A \wedge B}\;(\wedge\text{-E1}) \\
B
\end{array}
\qquad
\begin{array}{c}
(A \wedge B) \wedge C \\
C
\end{array}$$

$$B \wedge C$$

$$A \wedge (B \wedge C)$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\;(\wedge\text{-E1})}{A}\;(\wedge\text{-E1}) \qquad \dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\;(\wedge\text{-E1})}{B}\;(\wedge\text{-E2}) \qquad \begin{array}{c}(A \wedge B) \wedge C \\ \\ C\end{array}$$

$$B \wedge C$$

$$A \wedge (B \wedge C)$$

Proof example (Tree)

Prove: $(A \land B) \land C \vdash A \land (B \land C)$

$$\dfrac{\dfrac{(A \land B) \land C}{A \land B}\,(\land\text{-E1})}{A}\,(\land\text{-E1}) \qquad \dfrac{\dfrac{(A \land B) \land C}{A \land B}\,(\land\text{-E1})}{B}\,(\land\text{-E2}) \qquad \dfrac{(A \land B) \land C}{C}\,(\land\text{-E2})$$

$$B \land C$$

$$A \land (B \land C)$$

Proof example (Tree)

$$\text{Prove:} \quad (A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$$

$$\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{A}\,(\wedge\text{-E1}) \qquad \dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{B}\,(\wedge\text{-E2}) \qquad \dfrac{(A \wedge B) \wedge C}{C}\,(\wedge\text{-E2})}{B \wedge C}\,(\wedge\text{-I})$$

$$A \wedge (B \wedge C)$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$
\dfrac{
  \dfrac{
    \dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})
  }{A}\,(\wedge\text{-E1})
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})
    }{B}\,(\wedge\text{-E2})
    \qquad
    \dfrac{(A \wedge B) \wedge C}{C}\,(\wedge\text{-E2})
  }{B \wedge C}\,(\wedge\text{-I})
}{A \wedge (B \wedge C)}\,(\wedge\text{-I})
$$

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	**Premises**	**Formula**	**Rule**	**References**
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	**Premises**	**Formula**	**Rule**	**References**
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10

Proof example

Prove: $A \lor (B \land C) \vdash (A \lor B) \land (A \lor C)$

Line	Premises	Formula	Rule	References
1		$A \lor (B \land C)$	Premise	
2		A	Premise	
3	2	$A \lor B$	$\lor$-I1	2
4	2	$A \lor C$	$\lor$-I1	2
5	2	$(A \lor B) \land (A \lor C)$	$\land$-I	3, 4
6		$(B \land C)$	Premise	
7	6	B	$\land$-E1	6
8	6	$A \lor B$	$\lor$-I2	7
9	6	C	$\land$-E2	6
10	6	$A \lor C$	$\lor$-I2	9
11	6	$(A \lor B) \land (A \lor C)$	$\land$-I	8, 10
12	1	$(A \lor B) \land (A \lor C)$	$\lor$-E	5, 11

Proof example (Fitch)

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

1. A ∨ (B ∧ C)	
2. A	
3. A ∨ B	∨-I1: 2
4. A ∨ C	∨-I1: 2
5. (A ∨ B) ∧ (A ∨ C)	∧-I: 3,4

Proof example (Fitch)

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

1. $A \vee (B \wedge C)$	
2. A	
3. $A \vee B$	$\vee$-I1: 2
4. $A \vee C$	$\vee$-I1: 2
5. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 3,4
6. $B \wedge C$	
7. B	$\wedge$-E1: 6
8. $A \vee B$	$\vee$-I2: 7
9. C	$\wedge$-E2: 6
10. $A \vee C$	$\vee$-I2: 9
11. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 8,10

Proof example (Fitch)

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

1. $A \vee (B \wedge C)$	
2. A	
3. $A \vee B$	$\vee$-I1: 2
4. $A \vee C$	$\vee$-I1: 2
5. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 3,4
6. $B \wedge C$	
7. B	$\wedge$-E1: 6
8. $A \vee B$	$\vee$-I2: 7
9. C	$\wedge$-E2: 6
10. $A \vee C$	$\vee$-I2: 9
11. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 8,10
12. $(A \vee B) \wedge (A \vee C)$	$\vee$-E: 1,3–5,6–11

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	
2		$\neg A$	Premise	

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\bot$	$\neg$-E	$1, 2$

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\bot$	$\neg$-E	$1, 2$
4	1	$\neg\neg A$	$\neg$-I	3

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\bot$	$\neg$-E	$1, 2$

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\perp$	$\neg$-E	$1, 2$
4			?	

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\perp$	$\neg$-E	$1, 2$
4	1	A	IP	3